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THE OPTIMAL CONTROL SYNTHESIS OF THE SWINGING AND DAMPING OF A DOUBLE PENDULUM†

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The problem of constructing a control, which maximizes or minimizes the deviation of the first (upper) section of a plane double pendulum in one or several half-cycles of oscillations is solved. The angle between the sections, which can be varied within specified limits, is considered as the control parameter. Both this angle and its derivative occur in the equations of motion of the system, which complicates the solution of the problem. It is possible, by replacement of the variables, to eliminate the derivative of the control from the equations. After this, the optimal control is constructed in the form of feedback. The results of numerical investigations are presented. © 2001 Elsevier Science Ltd. All rights reserved.

The problems of controlling oscillatory system has attracted the attention of many researchers. They are investigated, for example, in the monographs [1–4]. Considerable difficulties usually arise in constructing the control, especially the optimal control, of objects in which the number of control actions is less than the number of degrees of freedom (objects with a "deficit" of the number of control parameters). Such objects include swings, pendulum systems, and walking mechanisms, in which drives are placed only at the hinges between the sections. Animals and humans can also displace "sections" of their bodies, but only one with respect to another. However, they do this so that the external forces which arise during relative motion – forces of interaction with the surroundings and gravity forces-move the body as a whole in the desired way. For example, walking, the running of animals, and the crawling of reptiles occurs as a result of friction forces from the supporting surface. Animals "organize" these forces in an appropriate way during the relative motion of the parts of the body. A person controls the oscillations of swings about the suspension point by displacing himself appropriately. There is no control moment at the suspension point. A gymnast swings on a horizontal bar by controlling mainly the angle at the hip joint. The moment at the wrist joint is extremely small in this case. In both of the latter cases, the person uses the gravity force in an appropriate way.

The problem of the optimal control of the swinging and damping of swings was investigated in [5]. The control parameter, as previously [2–4], was assumed to be the position of a point mass situated on them. The equations of motion obtained using the angle and angular velocity of the deflection of the swings from the vertical as the phase variables contain both the position and velocity of the point mass, in other words, contain the control together with its derivative, which is inconvenient for solving the problem of synthesizing the optimal control. The use of the angular momentum instead of the angular velocity as one of the phase variables enables one to eliminate the rate of displacement of the point mass from the equations of motion and to construct a complete picture of the optimal control synthesis.

Two problems of the optimal control of the swinging of a double physical pendulum have been investigated.‡ In the first problem, the rate of change of the angle between the sections, which is assumed to be bounded, is chosen as the control function; this problem is solved using the Pontryagin maximum principle. In the second problem, the angle between the sections itself is chosen as the control. In this case, the derivative of the angle is not eliminated from the equations of motion, and the necessary optimality conditions, by means of which numerical investigations are carried out, are obtained for this problem. It is not possible to construct the optimal control in the form of feedback (to solve the synthesis problem) without eliminating the derivative of the angle.

In this paper, when investigating the oscillations of a double pendulum, the angle between the sections is taken to be the control parameter, as in the second of the problems mentioned above. Unlike the equations of motion of swings, the use of the angular momentum and the angle of deviation from the vertical of the upper section of the double pendulum as the phase variables enables us to eliminate only the second derivative of the control parameter from its equations of motion. The first derivative

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can also be eliminated by introducing, instead of the angle of deviation of the upper section, a certain new variable, namely, the reduced angle of deviation. After this the problem of synthesizing the optimal control both of the swinging and damping of the double pendulum is solved without using the Pontryagin maximum principle.

The problem investigated here is of interest both from the point of view of theoretical mechanics and also for modelling the motions of a gymnast on a horizontal bar. Such motions have been investigated in a number of publications (see, for example, [6] and the bibliography given there).

1. THE EQUATIONS OF MOTION

Consider a plane double physical pendulum, suspended at the point O by an ideal (frictionless) cylindrical hinge (Fig. 1). The sections, which are absolutely rigid bodies, are also connected to one another by means of a cylindrical hinge at the point D. The axes of the hinges are horizontal and parallel to one another. The centre of mass of the first section will be assumed to be situated on the section OD. We will denote by $m_1^*, l_1^*, r_1^*, I_1^*$, the mass of the first (upper) section, its length OD, the distance of the point O to its centre of mass and its moment of inertia about the point O, respectively. Suppose $m_2^*, l_2^*, r_2^*, I_2^*$ are the similar quantities for the second section. We will denote the angle of deviation of the first section (segment OD) from the vertical, measured in an counter-clockwise direction, by φ, and we will denote the angle of deviation of the second section (the segment connecting the point D to the centre of mass of the second section) from the line passing through segment OD by α.

We will investigate an idealized control model in which the control parameter is the angle α between the sections, that varies within the following specified limits

$$\alpha_{\min} \leqslant \alpha \leqslant \alpha_{\max} \quad (\alpha_{\min}, \alpha_{\max} = \text{const}, \quad \alpha_{\min}, \alpha_{\max} \in (-\pi, \pi)) \tag{1.1}$$

We will assume that the permissible control is a piecewise-continuous function $\alpha(t)$ which pertains to the range (1.1). We will denote the set of permissible controls by U.

We will introduce the following dimensionless parameters of the double pendulum: the dimensionless time t and the angular momentum K (t^* and K^* are the corresponding dimensional quantities and g is the acceleration due to gravity)

$$r_1 = \frac{r_1^*}{l_1^*}, \quad I_1 = \frac{I_1^*}{m_1^* l_1^{*2}}, \quad r_2 = \frac{r_2^*}{l_1^*}, \quad m_2 = \frac{m_2^*}{m_1^*}, \quad I_2 = \frac{I_2^*}{m_1^* l_1^{*2}}; \quad t = t^* \sqrt{\frac{g}{l_1^*}}, \quad K = \frac{K^* \sqrt{l_1^*}}{m_1^* l_1^{*2} \sqrt{g}}$$

The expression for the (dimensionless) angular momentum K has the following form:

$$K = \xi(\alpha)\frac{d\varphi}{dt} + \eta(\alpha)\frac{d\alpha}{dt}; \quad \xi(\alpha) = I_1 + I_2 + m_2 + 2m_2 r_2 \cos\alpha, \quad \eta(\alpha) = I_2 + m_2 r_2 \cos\alpha \tag{1.2}$$

The expression $\xi(\alpha)$ describes the (dimensionless) moment of inertia of the double pendulum about the point O, and hence $\xi(\alpha) > 0$ for all values of the angle α.

The equations of motion of the system can be written in the form

$$dK/dt = \Phi(\varphi, \alpha), \quad \Phi(\varphi, \alpha) = -r_1 \sin\varphi - m_2[\sin\varphi + r_2 \sin(\varphi + \alpha)] \tag{1.3}$$

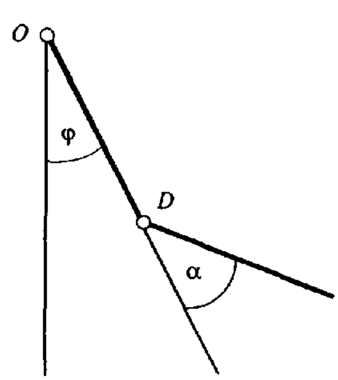

Fig. 1

$$\frac{d\varphi}{dt} = -\zeta(\alpha)\frac{d\alpha}{dt} + \frac{K}{\xi(\alpha)}, \quad \zeta(\alpha) = \frac{\eta(\alpha)}{\xi(\alpha)} \tag{1.4}$$

Equation (1.3) describes the change in the angular momentum of the system due to the effect of the moment of the gravity forces. Equation (1.4) is obtained by expanding relation (1.2) with respect to the derivative $d\varphi/dt$.

When using the angle φ and α and the angular velocities $d\varphi/dt$ and $d\alpha/dt$ as the phase variables of the system, higher derivatives $d^2\varphi/dt^2$ and $d^2\alpha/dt^2$ occur in the equation of motion. By considering the angular momentum K as one of the variables we can eliminate these higher derivatives, but the derivative $d\alpha/dt$ remains.

In model (1.3), (1.4) we have ignored the friction at the hinge O and the resistance of the surrounding medium to the motion of the pendulum.

2. A NEW VARIABLE – THE REDUCED ANGLE

Equations (1.3) and (1.4), in addition to the angle α, also contains its derivative $d\alpha/dt$. This makes it difficult to use the angle α as the control parameter and to solve the problem of synthesizing the optimal control, which is formulated below in Section 3.

Note that

$$\frac{d\varphi}{dt} + \zeta(\alpha)\frac{d\alpha}{dt} = \frac{d(\varphi - F(\alpha))}{dt} \tag{2.1}$$

where

$$F(\alpha) = -\frac{\alpha}{2} + A\operatorname{arctg}\left(B\operatorname{tg}\frac{\alpha}{2}\right) \tag{2.2}$$

The constant quantities A and B are defined as follows:

$$A = \frac{I_1 - I_2 + m_2}{R}, \quad B = \frac{R}{I_1 + I_2 + m_2 + 2m_2r_2}; \quad R = \sqrt{(I_1 + I_2 + m_2)^2 - 4m_2^2r_2^2}$$

In order to get rid of the derivative $d\alpha/dt$, we will introduce, instead of the angle φ, the reduced angle p, given by the formula (see Eq. (2.1))

$$p = \varphi - F(\alpha) \tag{2.3}$$

The variable (2.3) is introduced so that Eq. (1.4) can be written in the form

$$\frac{dp}{dt} = \frac{K}{\xi(\alpha)} \tag{2.4}$$

A conversion of the variable similar to (2.3) was considered previously in [7].

Thus, we will henceforth consider Eqs (1.3) and (2.4) in the variables K and p. Here we will assume that, instead of the angle φ, Eq. (1.3) contains a quantity calculated from (2.3) by the formula

$$\varphi = p + F(\alpha) \tag{2.5}$$

Note that if there is an abrupt change in the angle α, the angular momentum K of the system does not change (see Eq. (1.3)), like the reduced angle p (see Eq. (2.4)). If there is a sudden change in the angle α, there will be a sudden change in the angle φ, which is calculated using relation (2.5) (see also the paper cited in the footnote).

3. FORMULATION OF THE PROBLEM

Suppose we are given the initial state of system (1.3), (2.4)

$$p(0) < 0, \quad K(0) = 0 \tag{3.1}$$

If, for example, $\alpha(0) = 0$ (both sections lie in one straight line), then, with the initial condition (3.1), as follows from (2.5) the angle $\varphi(0) < 0$.

We will now formulate the problem of the optimum swinging of the double pendulum: it is required to obtain the law of variation of the control parameter α in the range (1.1), for which max $[p(T)]$ is reached, where T is the first instant of time after the motion begins (at $t = 0$) when $K(T) = 0$. We will write this formulation of the problem as follows:

$$\max\,[p(T)],\quad K(T) = 0,\quad T > 0 \tag{3.2}$$

Here and everywhere henceforth the operations max (maximization) and min (minimization) are carried out over all values of α in the specified limits (1.1).

This problem represents a problem of maximizing the reduced angle p in a "half-cycle" of the pendulum swing.

We will formulate the problem of the optimal braking of the pendulum in a half-cycle with condition (3.1) as follows:

$$\min\,[p(T)],\quad K(T) = 0,\quad T > 0 \tag{3.3}$$

When formulating problem (3.2) (problem (3.3)) the initial state (3.1) and the range (1.1) of permissible values of the control are assumed to be such that an instant of time $T > 0$ exists for which the angular momentum $K(T)$ vanishes, and a maximum (minimum) of the angle p is obtained.

Consider the function $F(\alpha)$ (2.2) and denote by α^* its maximizing argument in the range (1.1), and by α_* its minimizing argument:

$$\alpha^* = \arg\max F(\alpha),\quad \alpha_* = \arg\max \mathrm{F}(\alpha) \tag{3.4}$$

An extremum of the function $F(\alpha)$ is reached at one of the boundary points of the range (1.1) or inside it.

We will assume that the control $\alpha^0(t)$, which solves problem (3.2), has been obtained. We will also assume that at the end of the control interval $\alpha^0(t)$ – at the instant of time $t = T$ – the angle α is equal to α^* (3.4), which maximizes the function $F(\alpha)$. When there is an instantaneous change in the angle α, the angle p remains unchanged, while the angle φ changes abruptly and takes a value given by (2.5). It can be shown that the equation

$$\alpha = \alpha^0(t) \text{ when } 0 \leqslant t < T,\quad \alpha = \alpha^* \text{ when } t = T \tag{3.5}$$

maximizes the angle $\varphi(T)$, i.e. the swing of the first (upper) section of the double pendulum is a maximum in the half-cycle.

Suppose $\alpha_0(t)$ is the control which solves problem (3.3); then the equation

$$\alpha = \alpha_0(t) \text{ when } 0 \leqslant t < T,\quad \alpha = \alpha_* \text{ when } t = T \tag{3.6}$$

minimizes the angle $\varphi(T)$, i.e. the first (upper) section of the double pendulum receives the maximum damping in the half-cycle.

The functions (3.5) and (3.6), generally speaking, have a discontinuity at the point $t = T$. If $\alpha^* = \alpha^0(T)$ ($\alpha_* = \alpha_0(T)$), the function (3.5) (the function (3.6)) at this point remains continuous.

We will now describe a method of constructing the optimal control $\alpha^0(t)$, which solves problem (3.2), and the control $\alpha_0(t)$, which solves problem (3.3).

4. THE METHOD OF SOLVING THE PROBLEM

The quantity $\xi(\alpha)$ on the right-hand side of Eq. (2.4) is positive for any value of the angle α. Hence, the direction, in which the angle p changes (increases or decreases) depends only on the sign of the angular momentum K. The direction in which the angular momentum K changes depends, in turn, on the sign of the moment of the gravitational forces described (in dimensionless form) on the right-hand side of Eq. (1.3).

Suppose the initial state (3.1) is such that for all values of $\alpha \in U$ the moment of the gravitational forces when $t = 0$ is positive. Then, for any control $\alpha(t) \in U$ the value of K becomes positive at the beginning of the motion. We will assume that the angular momentum K remains positive up to a certain instant of time θ, when it again vanishes. Each trajectory in the control $\alpha(t) \in U$ has its own moment θ. The angle p, as follows from Eq. (2.4), increases strictly monotonically as the time t increases from 0 to θ.

In the (p, K) plane consider the set V of phase trajectories of system (1.3), (2.4), obtained for all $\alpha(t) \in U$ and which, at $t = 0$, emerge from the initial point (3.1) and which end at the instant $t = \theta$ (each on its own trajectory) on the $K = 0$ axis. The whose set V – an integral funnel [8, 9] – is situated, by convention, in the upper half-plane ($K > 0$) of the (p, K) phase plane (Fig. 2). The derivative $K' = dK/dp$ is a continuous function of the angle α. Hence, it reaches its maximum and minimum values in the range (1.1) for any p and K. It is obvious that on each phase trajectory of system (1.3), (2.4)

$$\min K' \leqslant K' \leqslant \max K'$$

We will assume that the solution of system (1.3)(2.4), at each point of which we have the equality

$$K' = \max K' \tag{4.1}$$

exists and the control corresponding to it is $\alpha(t) \in U$. Then, the corresponding phase trajectory belongs to the set V, and no other trajectory from V can intersect it from bottom to top. Consequently, the trajectory on which Eq. (4.1) holds is the upper boundary of the integral funnel V. It can be shown that during the motion along this upper boundary trajectory, a maximum of p at a finite instant of time is reached (when the angular momentum K vanishes). Consequently, this trajectory is the optimum one for problem (3.2).

The lower boundary of the integral funnel V is a trajectory (if such exists) of system (1.3), (2.4), at each point of which the derivative K' takes a minimum value with respect to the variable α

$$K' = \min K' \tag{4.2}$$

During the motion along this trajectory, a minimum of the angle p at the instant when the angular momentum K vanishes is reached, and it is the optimal one for problem (3.3).

5. SYNTHESIS OF THE OPTIMAL CONTROL

We conclude from expression (4.1) and Eqs (1.3) and (2.4) that the optimal control α^0, which solves problem (3.2), is given by the formula

$$\alpha^0 = \arg\max K' = \arg\max\,[\Phi(\varphi, \alpha)\xi(\alpha)] \tag{5.1}$$

The optimal control α_0, which solves problem (3.3), is given by the expression (see formula (4.2))

$$\alpha_0 = \arg\min K' = \arg\min\,[\Phi(\varphi, \alpha)\xi(\alpha)] \tag{5.2}$$

Before seeking the extrema, we must substitute into (5.1) and (5.2), instead of the angle φ, its expression (2.5) in terms of p and α. Formulae (5.1) and (5.2) solve the problem of synthesizing the optimal control. They define the quantities α^0 and α_0 in the form of functions of the reduced angle p

$$\alpha^0 = \alpha^0(p), \quad \alpha_0 = \alpha_0(p) \tag{5.3}$$

The quantities α^0 and α_0 obviously do not depend on the angular momentum K.

Whereas in the problem of the optimal control of swings, the extrema of the corresponding derivatives are obtained analytically [5], in this paper they can only be found numerically.

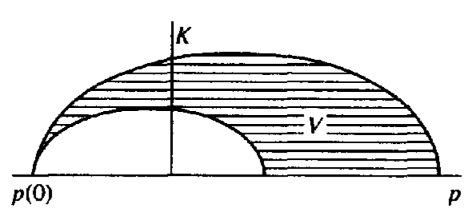

Fig. 2

We will note some properties of the control $\alpha^0 = \alpha^0(p)$ (5.1) and of the trajectories corresponding to it.

1. If the trajectory of system (1.3), (2.4) with control (5.1) falls on the axis $K = 0$, when $T > 0$, it also obviously solves problem (3.2) in the case when not all the integral funnel belongs to the upper half-plane of the phase plane.

2. Suppose $(\tilde{p}, \tilde{K})$ is an arbitrary point belonging to the trajectory of system (1.3), (2.4) and (5.1). Then, at the point of intersection of any other trajectory with the straight line $p = \tilde{p}$ the value of $K \leqslant \tilde{K}$.

3. Suppose that, for a certain initial state (3.1) the angular momentum K nowhere vanishes on the trajectory of system (1.3), (2.4) and (5.1) (the pendulum, without stopping, rotates all the time on one side). The value of K, reached on this trajectory for each specified value of p, is then the maximum possible.

We will consider the case when in the initial state $p(0) > 0, K(0) = 0$. If, under these conditions, the integral funnel is situated as a whole in the lower half-plane of the phase plane, then, using the same arguments as in Section 4, it can be shown that in this case also the optimal control which solves the swinging problem satisfies condition (4.1), and the optimal control which solves the damping problem satisfies condition (4.2). However, since the trajectories now lie in the half-plane $K < 0$, control (5.2) corresponds to condition (4.1), while control (5.1) corresponds to condition (4.2). In other words, the controls (5.3) change places: the control $\alpha_0 = \alpha_0(t)$ becomes the swinging control and control $\alpha^0 = \alpha^0(p)$ becomes the damping control.

Hence, by switching from one control (5.3) to the other, we can swing or damp the double pendulum when it oscillates both from right to left and from left to right. We will consider a certain number l of half-cycles in each of which the optimal "swinging" control is used. We will assume that in each of them the integral funnel is situated as whole in one of the corresponding half-planes and $\alpha_{min} \leqslant 0 < \alpha_{max}$. It can then be shown that the quantity $|p|$ takes the maximum possible value after l half-cycles. In order to maximize the quantity $|\varphi|$ after l half-cycles, we must put $\alpha = \alpha^*$ or $\alpha = \alpha_*$ at the last instant of time – at the end of the last half-cycle. It makes no sense to put $\alpha = \alpha^*$ or $\alpha = \alpha_*$ at the end of each half-cycle. When it swings the pendulum can, however, transfer into a state of unceasing rotation on one side, similar to the rotation of a gymnast carrying out a "sun" exercise.

6. NUMERICAL INVESTIGATIONS

The numerical investigations described here were carried out with the following "anthropomorphic" values of the parameters [10, 11]:

$$m_1^* = 38.4 \text{ kg}, \quad l_1^* = 1.19\text{m}, \quad r_1^* = 0.77\text{m}, \quad I_1^* = 28.72 \text{ kg m}^2$$
$$m_2^* = 26 \text{ kg}, \quad l_2^* = 1\text{m}, \quad r_2^* = 0.415\text{m}, \quad I_2^* = 6.3 \text{ kg m}^2 \qquad (6.1)$$

These parameters were calculated on the assumption that the first (upper) section is an absolutely rigid body consisting of the massive body of a human (a gymnast) and straightened arms. The lower section is assumed to be the legs.

We will put $\alpha_{min} = -2\pi/3$ and $\alpha_{min} = 0$. With these limiting values of the control parameter α, we are modelling the situation when the gymnast cannot bend backwards and cannot "fold' completely – his total folding remains 60°. Hence, in the numerical investigations we assume that

$$-2\pi/3 \leqslant \alpha \leqslant 0 \qquad (6.2)$$

Calculations show that for values of the parameters given by (6.1) the function (2.2) reaches a maximum and a minimum at the ends of the range (6.2)

$$\alpha^* = \alpha_{min} = -2\pi/3, \quad \alpha_* = \alpha_{max} = 0$$

Programs were written for solving Eqs (1.3) and (2.4) with control (5.1) and (5.2). Graphs obtained using these programs for the values of the parameters given by (6.1) are shown in Figs 3–8.

In Fig. 3 we show the optimal control $\alpha(t)$ which solves the problem of the optimal swinging of the double pendulum after three half-cycles, and the functions $p(t)$, $K(t)$, and $\varphi(t)$ corresponding to it. The solution shown was constructed for the initial conditions

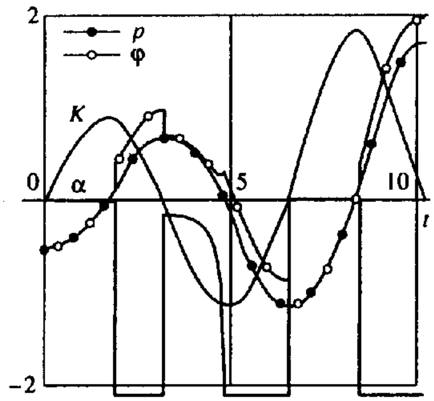

Fig. 3

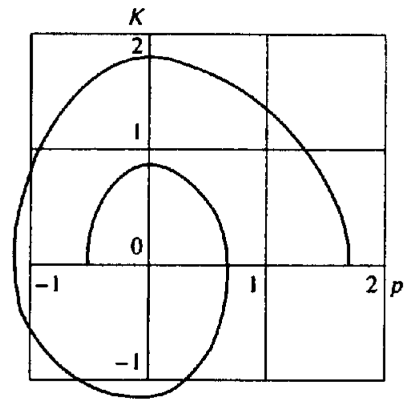

Fig. 4

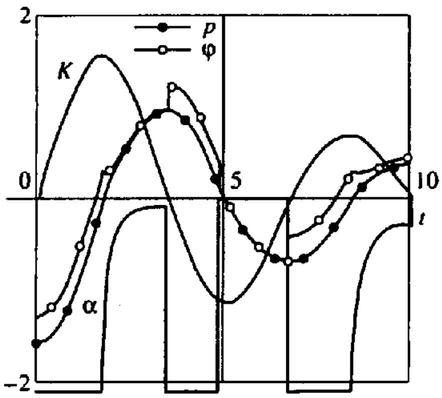

Fig. 5

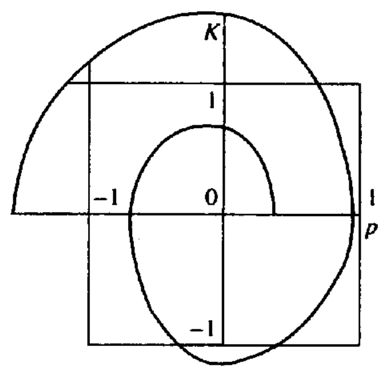

Fig. 6

$$p(0) = -\pi/6, \quad K(0) = 0$$

(since $\alpha(+0) = 0$ then also $\varphi(+0) = -\pi/6$). The control $\alpha = \alpha^0(t)$, when $K > 0$, and $\alpha = \alpha_0(t)$ when $K < 0$. In the first and third half-cycles (when the pendulum moves from left to right) the optimal control is $\alpha = \alpha^0(t)$, it takes only the limiting values and suffers discontinuities approximately when the angular momentum reaches a local maximum or minimum. The control $\alpha(t)$ also has a discontinuity, of course, when the kinetic moment passes through zero. In the second half-cycle (when the pendulum moves from right to left) the optimal control is $\alpha = \alpha_0(t)$, its does not always take its limiting values and changes continuously. The differences in the form of the control when the pendulum moves from one side to the other are due, in particular, to the fact that the limitations imposed on the control are asymmetrical. On the sections of the motion when $\alpha(t) = 0$, we have the equality $p(t) = \varphi(t)$, which agrees with Eq. (2.5). Along the other parts the angle p differs from the angle φ but is "close" to it. The angle φ has jumps at the instant when the control $\alpha(t)$ has a discontinuity. The time when the third half-cycle is completed is $t = \theta = 10.21$ (dimensionless time). The optimal control at the end of the third half-cycle $\alpha(\theta) = \alpha^* = -2\pi/3$, and hence the angle $\varphi(\theta) = 1.975$ is the maximum possible value after three half-cycles ($p(\theta) = 1.698$). In Fig. 4 the trajectory of the optimal swinging of the pendulum during three half-cycles is shown in the (p, K) phase plane.

In Fig. 5 we show the optimal control $\alpha(t)$, which solves the problem of the optimal damping of the double pendulum after three half-cycles, and the corresponding functions $p(t), K(t)$ and $\varphi(t)$. The solution is obtained for the initial conditions

$$p(0) = -\pi/2, \quad K(0) = 0$$

and constraints (6.2). The optimal control is $\alpha = \alpha_0(t)$, when $K > 0$, and $\alpha = \alpha^0(t)$, when $K < 0$. As follows from a consideration of Fig. 5

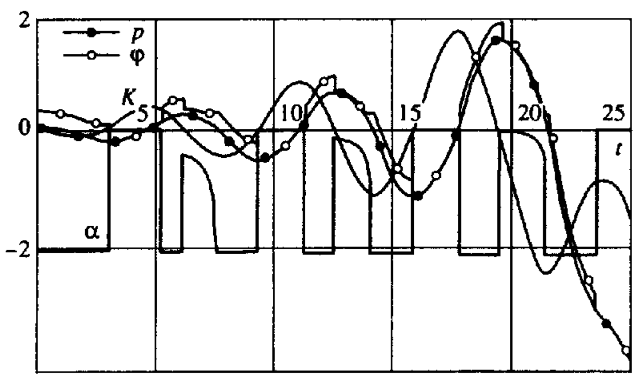

Fig. 7

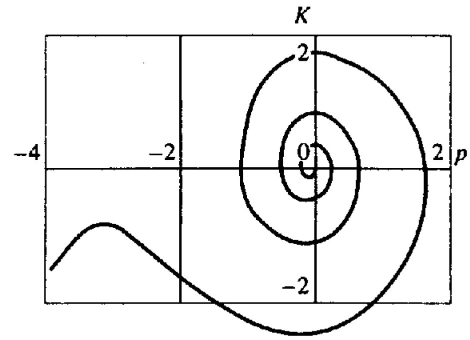

Fig. 8

$$\alpha(+0) = \alpha^* = -2\pi/3, \quad \varphi(+0) = p(0) + F(\alpha^*) = -\pi/2 + 0.2767 = -1.294$$

The control is $\alpha = \alpha_0(t)$ in the first and third half-cycles (when the pendulum moves from left to right), and in certain time intervals it changes continuously, not always taking its limiting values. In the second half-cycle (when the pendulum moves from right to left) the optimal control is $\alpha = \alpha^0(t)$, it takes only limiting values and has discontinuities approximately when the angular momentum reaches a local maximum or minimum. The control $\alpha(t)$ also has a discontinuity when the angular momentum vanishes. When $\alpha(t) = 0$, we have $p(t) = \varphi(t)$, and at the other time the angles p and φ are not equal to one another, but are "close". The time when the third half-cycle is completed $t = \theta = 10.06$. The control at the end of the third half-cycle $\alpha(\theta - 0) = -0.2827$ and is not equal to α_*, since $\alpha_* = 0$. Here the angle $\varphi(\theta - 0) = 0.4315$ is not the minimum possible value. If, at the instant of time $t = \theta$, the control angle α changes abruptly to a value $\alpha = \alpha_* = 0$, the angle φ will change abruptly. In this case after three half-cycles it takes the minimum possible value $\varphi(\theta) = p(\theta) = 0.3763$. In Fig. 5 we see these abrupt changes in the angles α and θ at the last instant of time $t = \theta = 10.06$. In Fig. 6 we show the trajectory of the optimal damping in the (p, K) phase plane.

In Fig. 7 we show the functions $\alpha(t)$, $K(t)$ and $\varphi(t)$, corresponding to the swinging of the pendulum with the initial conditions

$$p(0) = 0, \quad K(0) = 0$$

In $\alpha(0) = 0$, then $\varphi(0) = 0$, and in this case both sections of the pendulum drop vertically downwards and stay at rest at the initial instant of time. At the instant $t = +0$, the angle α abruptly takes the value $\alpha^* = -2\pi/3$, and simultaneously with this, in accordance with formula (2.5), the angle φ changes abruptly and takes its maximum value of 0.2767 (see Fig. 7). The double pendulum then swings, performing several oscillations to one side and the other with the optimal control $\alpha = \alpha^0(t)$, if $K > 0$, and $\alpha = \alpha_0(t)$, if $K < 0$. After several such swings the pendulum changes to a state of constant rotation similar to that of a gymnast carrying out a "sun" exercise. In this rotation $K < 0$ and $\alpha = \alpha_0(t)$. In Fig. 8 we show the motion described above in the (p, K) phase plane.

7. DISPLACEMENT OF THE DOUBLE PENDULUM FROM THE LOW TO THE HIGH POSITION

We will assume that the value $\alpha = 0$ lies inside the range (1.1). In this case, $\alpha_{min} < 0$ and $\alpha_{min} > 0$. Using the optimal control for the swinging of a double pendulum, we construct the control by means of which it transfers from the low stable equilibrium position

$$\varphi = 0, \quad \alpha = 0 \quad (p = 0), \quad K = 0 \tag{7.1}$$

when both sections drop vertically downwards, to the upper unstable position

$$\varphi = \pm\pi, \quad \alpha = 0 \quad (p = \pm\pi), \quad K = 0 \tag{7.2}$$

when both sections rise vertically upwards.

The pendulum can transfer from position (7.1) to position (7.2) in a finite time, for example, as follows. We will denote by W the potential energy of the pendulum when $p = -\pi$, $\alpha = \alpha_{min}$ ($\varphi = -\pi + F(\alpha_{min})$), $K = 0$. Beginning from state (7.1), we will swing the pendulum as shown in Figs 7 and 8, using control $\alpha_0(t)$ and $\alpha^0(t)$ alternately. During the swings, at each actual instant of time t, we calculate what the total energy (kinetic plus potential) of the pendulum would be if we put the control angle α equal to α_{min} at this instant of time t and kept it there. The pendulum in the "frozen" state, with $\alpha = \alpha_{min}$, rotates solely under the influence of the gravity force, like a simple pendulum. The total energy of this simple pendulum, which is preserved throughout the motion, is made of potential and kinetic energy. The latter is proportional to the square of the angular momentum K of a simple pendulum. Since, when there is a sudden change in the angle α at the instant of time t, the angular momentum K of the system remains the same, the total energy of this simple pendulum is easily calculated. Numerical investigations show that this calculated total energy of pendulum during the time t increases without limit, which is natural, since there are no damping forces in the system. As soon as this calculated energy becomes equal to W, we put $\alpha = \alpha_{min}$. Suppose then that the "frozen" pendulum (the simple pendulum) when $\alpha = \alpha_{min}$ rotates until its angular momentum K vanishes or, more accurately, until the reduced angle p becomes equal to $-\pi$. This equality is reached in a finite time. By changing the angle α from the value α_{min} to zero abruptly at this instant of time, we obtain $\varphi = -\pi$, in accordance with formula (2.5). The angular momentum K remains equal to zero during this sudden change. Hence, the pendulum is transferred to the desired upper unstable equilibrium position in a finite time. The algorithm for controlling the displacement of the double pendulum from the lower equilibrium position to the upper position, described above, is realized using a program.

The double pendulum can be transferred in a similar way to the upper equilibrium position with $\varphi = +\pi$ using the control $\alpha = \alpha_{max}$ instead of $\alpha = \alpha_{min}$.

The problem of stabilizing the upper unstable equilibrium position can be solved, for example, using linear of the form $\alpha = c_1 p + c_2 K$ with constant coefficients c_1 and c_2.

8. CONCLUSION

It is possible to solve the problem of synthesizing the optimal control of the oscillations of a double pendulum using the approach proposed in this paper. In a numerical investigation of any motion of the pendulum it is not necessary to solve the boundary-value problem which arises, for example, when using the Pontryagin maximum principle. When using formulae (5.1) and (5.2), describing the feedback, one only needs to solve the Cauchy problem, whereas to solve the boundary-value problem one needs to set up an iteration process. Hence, formulae (5.1) and (5.2) easily enable one to construct a numerical solution for any initial conditions and values of the pendulum parameters. Programs, written to solve Eqs (1.3) and (2.4) with controls (5.1) and (5.2) enable one to achieve animation.

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